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On the Minimum Roots of the Adjoint Polynomials of Unicyclic Graphs

Ren Haizhen[1,2] *Gao Yanling*[2]

(1. School of Mathematical Sciences, Xiamen University, Xiamen Fujian 361005;

2. Department of Mathematics, Qinghai Normal University, Xining Qinghai 810008)

Abstract The adjoint polynomial was introduced for solving the chromaticity problem of the complements of graphs. The minimum roots of the adjoint polynomials of graphs can be applied to sort out graphs that are not chromatically equivalent. Let $\beta(G)$ be the minimum root of the adjoint polynomial of the graph G. Denote by Ω_n the set of all unicyclic graphs on n vertices. All graphs with $max\{\beta(G)|G \in \Omega_n\}$(resp. $min\{\beta(G)|G \in \Omega_n\}$) are determined.

Key words Chromatic polynomial; Adjoint polynomial; Unicyclic graphs; Roots

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1 Introduction

All the graphs considered here are finite, undirected and simple. Undefined notation and terminology will refer to those in [1]. Let G be a graph, and $\overline{G}$, $V(G)$ and $E(G)$, respectively, be the complement, the vertex set and the edge set of G. The chromatic polynomial of G is defined as[2]

$$P(G,\lambda) = \sum_{r=1}^{|V(G)|} m_r(G)(\lambda)_r,$$

where $m_r(G)$ denotes the number of r-independent partitions in G and $(\lambda)_i = \lambda(\lambda - 1)\cdots(\lambda - i + 1)$ for all $i \geq 1$. And the adjoint polynomial of G is defined as

$$h(G,x) = \sum_{i=0}^{|V(G)|-1} m_{|V(G)|-i}(\overline{G})x^{|V(G)|-i}.$$

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Note that the adjoint polynomial of G has real roots. Denote by $\beta(h(G,x))$ the minimum real root of $h(G,x)$. For brevity we shall write $h(G)$ instead of $h(G,x)$, and $\beta(G)$ instead of $\beta(h(G,x))$. Two graphs G and H are chromatically equivalent if $P(G,\lambda) = P(H,\lambda)$, in notation $G \overset{P}{\sim} H$. Similarly, two graphs G and H are adjointly equivalent if $h(G,x) = h(H,x)$, in notation $G \overset{h}{\sim} H$. It is obvious that $\overline{G} \overset{P}{\sim} \overline{H}$ if and only if $G \overset{h}{\sim} H$.

George David Birkhoff introduced the chromatic polynomial in 1912, defining it only for planar graphs, in an attempt to prove the four color theorem. In 1968 Read asked which polynomials are the chromatic polynomials of some graph, a question that remains open, and introduced the concept of chromatically equivalent graphs[2]. Today, chromatic polynomials are one of the central objects of algebraic graph theory. The adjoint polynomial was introduced for solving the chromaticity problem of the complements of graphs[3,4]. For a survey of the mathematical properties and related results see the review [4]. Let $\beta(G)$ be the minimum root of the adjoint polynomial of the graph G. The minimum roots of the adjoint polynomials of graphs can be applied to sort out graphs that are not chromatically equivalent. In [5 − 8], the authors studied the minimum real roots of the adjoint polynomials of some graphs, many new classes of chromatically unique(chromatically equivalent) graphs had been obtained by comparing the minimum roots of their adjoint polynomials. Thus, it is useful to find the extremal values of $\beta(\mathcal{G})$ for significant classes of graphs $\mathcal{G}$. A connected graph with n vertices is said to be unicyclic if it has n edges. Let Ω_n be the set of all unicyclic graphs on n vertices. In this paper, all graphs with $\max\{\beta(G)|G \in \Omega_n\}$, or just $\max_\beta(\Omega_n)$(resp. $\min\{\beta(G)|G \in \Omega_n\}$, or just $\min_\beta(\Omega_n)$) are determined.

2 Some Notations and Lammas

Let K_n, P_n and C_n be the complete graph, the path and the cycle of n vertices, respectively. Denote by D_n the graph obtained by identifying a vertex of C_3 with one end-vertex of P_{n-2}. For a vertex v of G, we denote by $N_G(v)$ the set of vertices of G which are adjacent to v. We denote $G - v$ the graph obtained from G by deleting the vertex v and edges incident to it. For two graphs G and H, $G \cup H$ denotes the disjoint union of G and H, and mH the disjoint union of m copies of

H. Let $f(x)$ be a polynomial in x, denote by $\partial(f(x))$ the degree of $f(x)$.

A simple x_1x_k-path of G is a path $x_1x_2\cdots x_k$(possibly $x_1=x_k$) of G such that $d(x_2)=d(x_3)=\cdots=d(x_{k-1})=2$(unless $k=2$), where $d(x_i)$ denotes the degree of the vertex x_i in G. Moreover, a simple x_1x_k-path of G is called an internal x_1x_k-path of G if $d(x_1)$ and $d(x_k)$ are at least 3. The following Lemmas will be used in sequel.

Lemma 1[4] If G has k connected components $G_1,G_2,\cdots,G_k$, then

$$h(G)=\Pi_{i=1}^{k}h(G_i).$$

For any edge $e=uv$ of a graph G, the graph $G*e$ is defined as the follows: the vertex set of $G*e$ is $V(G*e)=\{V(G)\setminus\{u,v\}\}\cup\{x\}$, and the edge set of $G*e$ is $E(G*e)=\{e\in E(G)|e \text{ is not incident with } u \text{ or } v\}\cup\{xy|y\in N_G(u)\cap N_G(v)\}$, where x does not belong to $V(G)$.

Lemma 2[4] For any $e\in E(G)$, we have $h(G)=h(G-e)+h(G*e)$, where $G-e$ denotes the graph obtained from G by deleting the edge e. In particular, if edge $e=uv$ does not belong to any triangle in G. Then

$$h(G,x)=h(G-uv,x)+xh(G-\{u,v\},x).$$

For convenience, we suppose that $h(P_{-1})=0$, $h(P_0)=1$ and $h(P_1)=x$. Then, for $n\geq 0$ we have

$$h(P_{n+1})=x[h(P_n)+h(P_{n-1})].$$

Lemma 3[7] Let G be a connected graph and H a proper subgraph of G. Then $\beta(G)<\beta(H)$.

Lemma 4[9] Let $f_i(x)$ be the real coefficient polynomials in the form $f_i(x)=\sum_{j=1}^{n_i}a_{ij}x^j$ such that $a_{in_i}>0$, where $i=1,2$ and n_i are positive integers. If $\beta_1\neq\beta_2$ and the following:

(1) $f_3(x)=f_2(x)+f_1(x)$ and $\partial_2-\partial_1\equiv 0(mod\ 2)$ or

(2) $f_3(x)=f_2(x)-f_1(x)$ and $\partial_2-\partial_1\equiv 1(mod\ 2)$

holds, then there exists at least one real root β_3 such that $\beta_3>min\{\beta_1,\beta_2\}$, where β_i denotes the minimum roots of $f_i(x)$ and ∂_i the degree of $f_i(x)$ for $i=1,2,3$.

Lemma 5[10] Let $H_m(G,P_{s+1},P_{t+1})$ be the graph with order m obtained from G by identifying u with an end-vertex of P_{s+1} (resp. P_{t+1}), where $m=n+s+t$, $n\geq 2$ and $1\leq s\leq t$. Then

$$\beta(H_m(G,P_{s+1},P_{t+1}))<\beta(H_m(G,P_s,P_{t+2})).$$

For a connected graph G, let $e = uv \in E(G)$. We carry out the following transformations on G. Contracting the edge $e = uv$(i.e. identifying u with v) and adding a pendent edge to the vertex $u(= v)$, the resulted graph is denoted by G_0(as shown in Figure 1(a)).

Lemma 6[11] Let G and G_0 be two graphs as shown in Figure 1(a). If u and v do not belong to any triangle in G, then $\beta(G) > \beta(G_0)$.

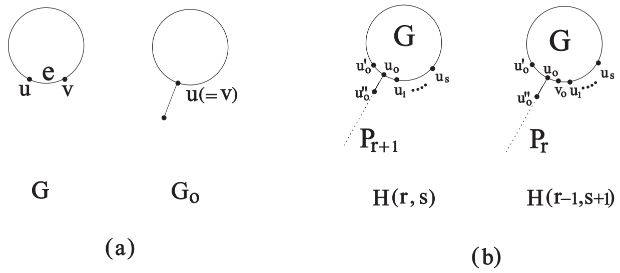

Figure 1: (a) G, G_0; (b) $H(r, s), H(r-1, s+1)$.

Let G be a graph and $e \in E(G)$. Insetting k vertices of degree 2 in e is said to be $k-$subdivision of an edge e. Let $u_0, u_1, \cdots, u_s (s \geq 1)$ be a simple path of graph G. Denote by $H(r, s)$ the graph obtained from G by identifying the vertex u_0 with an end-vertex of $P_{r+1} (r \geq 1)$. We carry out the following transformations on $H(r, s)$. Inserting a vertex v_0 in edge u_0u_1(i.e. carrying out 1−subdivision in edge u_0u_1) and deleting another end-vertex of P_{r+1}, the resulted graph is denoted by $H(r-1, s+1)$, as shown in Figure 1(b).

Lemma 7[11] Let $H(r, s)$ and $H(r-1, s+1)$ be two graphs as shown in Figure 1(b). Suppose that u_0u_s-path is an internal path. If $d(u_0) = 3$ and u_0 does not belong to any triangle in G. Then $\beta(H(r, s)) < \beta(H(r-1, s+1))$.

Lemma 8[12] Let G be a graph and $u \in V(G)$. Then $h(G) = x \sum\limits_{u \in V(K_j), j \geq 1} h(G - K_j)$, where the summation is over all the complete subgraphs of G which contain u.

3 Main Results

Theorem 1 Let $G \in \Omega_n$. Denote by E_n the graph obtained from K_3 by adding $n - 3$ pendent edges to one of vertices in K_3. Then (i) if $n = 4$ then $\beta(E_n) = \beta(G) = \beta(C_n)$; (ii) and if $n > 4$ then $\beta(E_n) \leq \beta(G) \leq \beta(C_n)$, with right-hand equality holds only if $G \cong C_n$, and left-hand equality holds if $G \cong E_n$.

Proof Since an unicyclic graph is either a cycle or a cycle with trees attached. When $n = 4$. Then $\Omega_4 = \{C_4, E_4\}$. By Lemmas 1–2, $h(C_4) = h(E_4) = h(P_4) + xh(P_2)$. Thus, $\beta(E_4) = \beta(G) = \beta(C_4)$. For $n > 4$, suppose that $G \ncong C_n$. Let C_m be the cycle of order m in G, where $m < n$; and let u_i be all vertices of C_m with $d(u_i) \geq 3$ in G, where $i = 1, 2, \cdots, t$. Applying Lemma 5 successively in turn with each tree hanging to vertex u_i in G, we obtain

$$\beta(C_m(u_1, \cdots, u_t)(r_1P_2, \cdots, r_tP_2)) \leq \beta(G) \leq \beta(C_m(u_1, \cdots, u_t)(P_{l_1}, \cdots, P_{l_t})),$$

where $C_m(u_1, \cdots, u_t)(P_{l_1}, \cdots, P_{l_t})$ is the graph of order $m+l_1+l_2+\cdots+l_t$ constructed by identifing each vertex u_i of C_m with an end-vertex of P_{l_i+1}(as shown in Figure 2(a)); and $C_m(u_1, \cdots, u_t)(r_1P_2, \cdots, r_tP_2)$ is the graph of order $m+r_1+r_2+\cdots+r_t$ constructed by adding r_i pendent edges to each vertex u_i of C_m(as shown in Figure 2(b)).

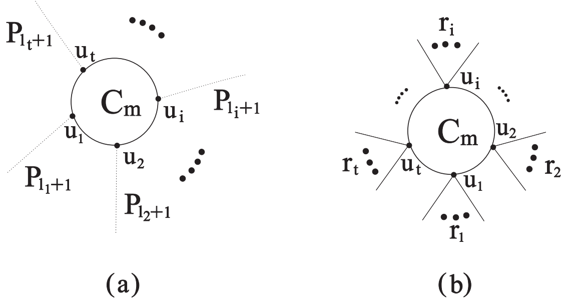

Figure 2: (a) $C_m(u_1, \cdots, u_t)(P_{l_1}, \cdots, P_{l_t})$; (b) $C_m(u_1, \cdots, u_t)(r_1P_2, \cdots, r_tP_2)$.

Now we prove that $\beta(C_m(u_1, \cdots, u_t)(P_{l_1}, \cdots, P_{l_t})) < \beta(C_n)$.

Case 1 If $t = 1$, applying Lemmas 1–2 to $C_m(u_1, \cdots, u_t)(P_{l_1}, \cdots, P_{l_t})$ we have $h(C_n) = h(P_n) + xh(P_{n-2})$ and

$$h(C_m(u_1)(P_{l_1})) = \begin{cases} h(P_n) + xh(P_{m-2})h(P_{l_1}), & \textit{if } m > 3, \\ h(P_n) + h(P_2)h(P_{l_1}), & \textit{if } m = 3. \end{cases}$$

Thus,

$$h(C_m(u_1)(P_{l_1})) - h(C_n) = \begin{cases} -x^2h(P_{m-3})h(P_{l_1-1}), & \textit{if } m > 3, \\ x^2h(P_{l_1-2}), & \textit{if } m = 3. \end{cases}$$

By Lemmas 3–4, we know that $\beta(C_m(u_1)(P_{l_1})) < \beta(C_n)$.

Case 2 If $t > 1$, applying Lemma 7 to the graph $C_m(u_1, \cdots, u_t)(P_{l_1}, \cdots, P_{l_t})$ we get $\beta(G) < \beta(C_{m+l_2+\cdots+l_t}(u_1)(P_{l_1}))$. From Case 1, we know that $\beta(C_{m+l_2+\cdots+l_t}(u_1)(P_{l_1})) < \beta(C_n)$. In the following, if $G \ncong E_n$, we prove that $\beta(C_m(u_1, \cdots, u_t)(r_1P_2, \cdots, r_tP_2)) > \beta(E_n)$.

Case 3 If $t = 1$, then $m > 3$. Applying Lemmas 6 to $C_m(u_1)(r_1P_2)$ we obtain $\beta(C_m(u_1)(r_1P_2)) > \beta(E_n)$.

Case 4 If $t > 1$, applying Lemma 6 to the graph $C_m(u_1, \cdots, u_t)(r_1P_2, \cdots, r_tP_2)$ we obtain $\beta(C_m(u_1, \cdots, u_t)(r_1P_2, \cdots, r_tP_2)) > \beta(C_3(u_1, u_2, u_3)(r_1'P_2, r_2'P_2, r_3'P_2))$, where $r_1' + r_2' + r_3' = n - 3$ and $r_1' \le r_2' \le r_3'$. By Lemma 8, we obtain

$$h(C_3(u_1, u_2, u_3)(r_1'P_2, r_2'P_2, r_3'P_2)) - h(C_3(u_1, u_2, u_3)((r_1' - 1)P_2, (r_2' + 1)P_2, r_3'P_2)) \\ = (r_2' - r_1' + 1)(x + r_3')x^{n-3}.$$

Thus, by Lemmas 3–4 we know that

$$\beta(C_3(u_1, u_2, u_3)(r_1'P_2, r_2'P_2, r_3'P_2)) > \beta(C_3(u_1, u_2, u_3)((r_1' - 1)P_2, (r_2' + 1)P_2, r_3'P_2)),$$

which implies that $\beta(C_3(u_1, u_2, u_3)(r_1'P_2, r_2'P_2, r_3'P_2)) > \beta(E_n)$. Therefore,

$$\beta(C_m(u_1, \cdots, u_t)(r_1P_2, \cdots, r_tP_2)) > \beta(E_n).$$

Theorem 1 holds.

From Theorem 1, we can deduce the following Corollary 1.

Corollary 1 Let $G \in \Omega_n$ and $n \ge 4$. If $n = 4$ then $max_\beta(\Omega_n)(= min_\beta(\Omega_n)) = \beta(C_n) = \beta(D_n)$. Otherwise, $max_\beta(\Omega_n) = \beta(C_n)$ and $min_\beta(\Omega_n) = \beta(E_n)$.

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单圈图的伴随多项式的极小根

任海珍[1,2] 高燕玲[2]

(1. 厦门大学数学科学学院, 厦门 福建 361005;

2. 青海师范大学数学系, 西宁 青海 810008)

摘 要 引入伴随多项式是为了从补图的角度研究色多形式, 图的伴随多项式的极小根可用于判定色等价图. $\beta(G)$表示图G的伴随多项式的极小根. Ω_n表示n个顶点的单圈图的集合. 分别确定了具有$max\{\beta(G)|G \in \Omega_n\}$和$min\{\beta(G)|G \in \Omega_n\}$ 的所有单圈图.

关键词 色多项式; 伴随多项式; 单圈图; 根